Approximation-Based Self-Triggered Model Predictive Control for Perturbed Nonlinear Systems

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Abstract— **This paper proposes an approximation-based self-triggered model predictive control strategy for nonlinear sampled-data systems with additive disturbance and system constraints. In our strategy, the finite horizon optimal control problem (FHOCP) and the triggering condition are designed based on approximate models in a discrete-time manner. By implementing the strategy, the computation problem of the FHOCP becomes tractable since it is computed in a discrete-time framework. Meanwhile, the next triggering instant is pre-determined by the triggering condition, reducing the sensing cost and the computing frequency of the FHOCP. Furthermore, feasibility of the FHOCP and stability of the overall system are analyzed. Finally, a simulation example verifies the effectiveness of the strategy.**

Index Terms— **Self-triggered control, approximations, model predictive control, sampled-data systems.**

I. Introduction

Over the past few decades, model predictive control (MPC) has been extensively applied to deal with the control problem of practical systems [1]. As an efficient technique in handling system constraints and optimizing control performance, MPC requires to solve online a finite horizon optimal control problem (FHOCP) to obtain a sequence of control actions [2]. In general, the MPC algorithms for continuous-time systems are required to be implemented under digital platforms. Such a combination of continuous- and discrete-time signals leads to the formation of "sampled-data" systems. However, solving the FHOCP at each sampling instant in sampled-data MPC will aggravate the computation load. This turns out to be the disadvantage of its application in a wide range of systems, to name a few, large-scale systems with huge total amount of computing tasks, systems which need fast response like unmanned vehicles, and systems with limited computing resources such as micro robots. Therefore designing more efficient MPC algorithms to reduce resource consuming has been an attracting problem currently.

One promising approach towards this problem is to introduce event-driven strategies into sampled-data MPC. In general, the event-triggered MPC (ETMPC) and self-triggered MPC (STMPC), which have been proposed in [3]–[8], are the two main cases. These strategies focus on reducing the computing frequency of the FHOCP to achieve a better trade-off between resource consuming and system performance.

In particular, the ETMPC checks whether the triggering conditions at each sampling instant are violated or not. Once violated, the computation of the FHOCP is activated. In view of this principle, it can be seen in [3] that an ETMPC strategy based on system stability is put forward by guaranteeing the Lyapunov function to be decreasing between the two successive triggering instants. While in [4], the triggering condition is derived based on recursive feasibility by bounding the deviation between the true states and the corresponding predicted ones. The above results is of great importance since unnecessary computation of optimal control problems are neglected. However, one may notice that the ETMPC requires constantly monitoring the system states, causing high sensing cost. To avoid this, the STMPC, which obtains pre-determined next triggering instant by applying the latest system states, has been widely studied. In [7], an STMPC strategy is studied for continuous-time perturbed nonlinear systems. In [8], a novel constraint tightening strategy is proposed and the prediction horizon is able to be decreased adaptively with the self-triggered scheme.

Nevertheless, previous works on event-driven MPC strategies mostly consider continuous- or discrete-time systems separately such that these strategies cannot be directly implemented in sampled-data systems. To overcome such difficulty, authors of [9] propose two event-based sampled-data MPC strategies by applying the non-monotonic Lyapunov method and introduce a tightened set of state constraint to guarantee the inter-sampling behavior. In [10], an approximation-based ETMPC strategy is proposed, with which the approximate discrete-time model and discretized cost function facilitate the computation of the control problem. However, results on self-triggered sampled-data MPC can still not be found in state-of-the-art research.

We propose, in this paper, a self-triggered strategy for sampled-data MPC to deal with general perturbed constrained sampled-data nonlinear systems. The main contributions of our work lie in:

1) A more applicable robust approximation-based STMPC strategy is put forward, in which the approximation-based method provides a solution for the unavailability of the exact model of the nonlinear system and the STMPC scheme is able to reduce the computation load effectively;
2) The sufficient conditions for recursive feasibility of the FHOCP and stability of the overall system are proposed. In particular, the feasibility and the stability rely on the model error and the upper bound of the external disturbance.

The rest of this paper is structured as follows: Section II gives preliminary setup for system description and problem formulation. Section III shows the main results of the self-triggered strategy and analyzes the feasibility and the stability. A simulation example in Section IV verifies the effectiveness of the strategy. Section V summarizes the paper.

Notations: The real and nonnegative integers are represented as $\mathbb{R}$ and $\mathbb{Z}$ respectively in this paper. The n-dimensional Euclidean space is denoted as $\mathbb{R}^n$. For a matrix Q, we denote its maximum and minimum eigenvalues as $\overline{\lambda}(Q)$ and $\underline{\lambda}(Q)$. A symmetric matrix P is positive definite if $P > 0$. And for a vector x, $\|x\| := \sqrt{x^T x}$ and $\|x\|_P := \sqrt{x^T P x}$ respectively represent its Euclidean norm and P-weighted norm. $\mathbf{s}_{[t_1,t_2]}$ is the truncation of a continuous signal $\mathbf{s}(t)$ from time t_1 to t_2. Define the Pontryagin difference set as $\mathcal{A} \sim \mathcal{B} := \{x : x + y \in \mathcal{A}, \forall y \in \mathcal{B}\}$ for two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$.

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II. Problem Setup

Consider the following perturbed continuous-time nonlinear system:

$$\dot{x}(t) = f(x(t), u(t)) + w(t), t \geq 0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent state and input of the system, respectively, and $w(t) \in \mathbb{R}^n$ stands for the external disturbance. It is assumed that they are subject to the constraints given by

$$x(t) \in \mathcal{X}, u(t) \in \mathcal{U}, w(t) \in \mathcal{W} \quad (2)$$

where $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$, and $\mathcal{W} := \{w \in \mathbb{R}^n : \|w\|_R \leq \eta\}$ are compact sets in which the origin is an interior point. R is a positive definite weighting matrix and η is a nonnegative constant. The solution of system (1) at time t is denoted as $\phi(t; x_0, \mathbf{u}, \mathbf{w})$ with initial state x_0, control input $\mathbf{u}$, and external disturbance $\mathbf{w}$. Furthermore, we assume that system (1) satisfies the property given below.

Assumption 1: The function $f(x, u)$ is locally Lipschitz continuous in x depending on the weighting matrix R with a Lipschitz constant $L > 0$:

$$\begin{aligned} \|f(x_1, u) - f(x_2, u)\|_R \leq L\|x_1 - x_2\|_R, \\ \forall x_1, x_2 \in \mathcal{X}, u \in \mathcal{U}. \end{aligned} \quad (3)$$

In general, the MPC algorithms need to be implemented under digital platforms. Therefore, we shall first define the exact discrete-time model (DTM) of system (1) as follows:

$$x(t_{k+1}) = \Gamma_T^e(x(t_k), u(t_k)) + w(t_k) \quad (4)$$

where the sampling instant is denoted by $t_{k+i} = (k+i)T$ in which T is the sampling period, i.e., $t_{k+1} = t_k + T$ with $k, i \in \mathbb{Z}$. $\Gamma_T^e(x, u)$ is the exact disturbance-free DTM of system (1) and $w(t_k) := \int_{t_k}^{t_{k+1}} w(s)ds$.

The standard MPC requires periodically solving an FHOCP at each sampling instant, aggravating the computation load. In view of this, our aim is to propose an STMPC strategy, in which the triggering instants of computing the FHOCP are determined in terms of the designed triggering condition.

Suppose that t_{k_j} denotes the j-th triggering instant with $j \in \mathbb{Z}$, i.e., t_{k_j} and $t_{k_{j+1}}$ are two successive triggering instants:

$$t_{k_{j+1}} = t_{k_j} + mT \quad (5)$$

where mT is the triggering interval determined by the triggering condition.

Note that when implementing STMPC strategy, one problem arises that the explicit expressions of $\Gamma_T^e(x, u)$ are not available for general nonlinear systems. Another problem is how to select a proper m. One can notice from (5) that lager triggering interval indicates lower frequency of computation. Therefore, the key of the STMPC strategy lies in two aspects: 1) adopting a system model with a simple expression; 2) designing a triggering condition to enlarge m while ensuring feasibility and stability.

III. Approximation-Based STMPC Strategy

A. The FHOCP with Approximate DTM

As mentioned above, the key to solve the first problem is to use a model which can be more easily obtained. Therefore, we define the approximate DTM of system (1) without disturbance as

$$\hat{x}(t_{k+i+1}|t_k) = \Gamma_T^a(\hat{x}(t_{k+i}|t_k), \hat{u}(t_{k+i}|t_k)) \quad (6)$$

where $t_{k+i} = t_k + iT$ and $\hat{x}(t_{k+i}|t_k)$ is the ith predicted state since $x(t_k)$. Here $\Gamma_T^a(x, u)$ can be obtained in different manners in terms of numerical methods. Moreover, Γ_T^e and Γ_T^a should satisfy the assumption given below.

Assumption 2: The function $\Gamma_T^a(x, u)$ is locally Lipschitz continuous in x depending on the weighting matrix R and the model error is bounded:

$$\|\Gamma_T^a(x, u) - \Gamma_T^a(y, u)\|_R \leq e^{LT}\|x - y\|_R \quad (7)$$

$$\|\Gamma_T^e(x, u) - \Gamma_T^a(x, u)\|_R \leq T\rho(T) \quad (8)$$

where ρ is a class-$\mathcal{K}_\infty$ function, see [11].

Remark 1: The above two inequalities are fairly standard assumptions for the exact and approximate DTM, see [11], [12]. The inequality (7), indicating that $\Gamma_T^a(x, u)$ is locally Lipschitz continuous, has a different form compared to inequality (3) due to the utilization of Gronwall-Bellman inequality. The inequality (8) bounds the model error between Γ_T^e and Γ_T^a over time interval $[t_{k+i}, t_{k+i+1}]$, and the model error can be reduced by applying more accurate approximation methods.

Based on the assumptions given above, we can derive the upper bound of the deviation between the real system (1) and its corresponding approximate DTM (6).

Lemma 1 ([10]): Denote the state error as: $\mathbf{e}_{[t_{k+i}, t_{k+i}+\tau]} = \phi(t_{k+i} + \tau; x(t_k), \mathbf{u}_{[t_{k+i}, t_{k+i}+\tau]}, \mathbf{w}_{[t_{k+i}, t_{k+i}+\tau]}) - \hat{x}(t_{k+i}|t_k)$, with $i = 0, 1, ..., N-1$ and $\tau \in [0, T)$. With Assumption 1 and 2 hold, the state error has an upper bound as

$$\|\mathbf{e}_{[t_{k+i}, t_{k+i}+\tau]}\|_R \leq \gamma\tau + \frac{e^{iLT} - 1}{e^{LT} - 1}(T\rho(T) + \eta T) \quad (9)$$

where the original system (1) is bounded by $\|f(x(t), u(t)) + w(t)\|_R \leq \gamma$.

With the lemma given above, by defining the Pontryagin difference set: $\mathcal{X}_i = \mathcal{X} \sim \mathcal{B}_i$ where $\mathcal{B}_i = \{x \in \mathbb{R}^n : \|x\|_R \leq \gamma\tau + \frac{e^{iLT}-1}{e^{LT}-1}(T\rho(T)+\eta T)\}$, one can notice that once the predicted states satisfy $\hat{x}(t_{k+i}|t_k) \in \mathcal{X}_i$ in the approximate DTM, the continuous-time states of system (1) over time interval $[t_{k+i}, t_{k+i} + \tau]$ will be guaranteed to meet the constraints in (2).

The cost function is defined as

$$\begin{aligned} &J(x(t_k), \hat{\mathbf{u}}(t_k), N) \\ &= \sum_{i=0}^{N-1} TV_l(\hat{x}(t_{k+i}|t_k), \hat{u}(t_{k+i}|t_k)) + V_f(\hat{x}(t_{k+N}|t_k)) \end{aligned} \quad (10)$$

where N stands for prediction horizon and $\hat{\mathbf{u}}(t_k) = \{\hat{u}(t_k|t_k), \hat{u}(t_{k+1}|t_k), ..., \hat{u}(t_{k+N-1}|t_k)\}$ is a sequence of control input. $V_l(x, u) = \|x\|_Q^2 + \|u\|_P^2$ is the stage cost, $V_f(x) = \|x\|_R^2$ is the terminal cost, Q, P, R are positive definite weighting matrices. With $i = 0, 1, ..., N-1$ and $\hat{x}(t_k|t_k) = x(t_k)$, the FHOCP based on approximate DTM is formulated as follows:

$$\begin{aligned} &\min_{\hat{\mathbf{u}}(k)} \ J(x(k), \hat{\mathbf{u}}(k), N) \\ s.t. \quad &\hat{x}(k+i+1|k) = \Gamma_T^a(\hat{x}(k+i|k), \hat{u}(k+i|k)), \\ &\hat{x}(k+i|k) \in \mathcal{X}_i, \\ &\hat{u}(k+i|k) \in \mathcal{U}, \\ &\hat{x}(k+N|k) \in \mathcal{X}_f \end{aligned} \quad (11)$$

where we denote t_{k+i} as $k+i$ for simplicity and $\mathcal{X}_f$ represents the terminal state constraint set: $\mathcal{X}_f = \{x \in \mathbb{R}^n : \|x\|_R \leq \varepsilon\}$.

Furthermore, the following assumption is stated to establish our self-triggered MPC strategy.

Assumption 3: The stage cost $V_l(x, u)$, the terminal cost $V_f(x)$, a local controller $\kappa(x)$, the terminal state constraint set $\mathcal{X}_f$, another significant set $\mathcal{X}_r$ satisfy the properties given below:

1) $0 \in \mathcal{X}_f$, $\mathcal{X}_f \subset \mathcal{X}_r = \{x \in \mathbb{R}^n : \|x\|_R \leq r\}$ with $0 < \varepsilon < r$ and $\mathcal{X}_r \subseteq \{x \in \mathcal{X}_{N-1} : \kappa(x) \in \mathcal{U}\}$;

2) $\Gamma_T^a(x,\kappa(x)) \in \mathcal{X}_f$, $\forall x \in \mathcal{X}_r$;
3) $V_f(\Gamma_T^a(x,\kappa(x))) - V_f(x) \le -TV_l(x,\kappa(x))$, $\forall x \in \mathcal{X}_r$;

***Remark** 2:* The above properties, which are applied as guidelines to determine $\kappa(x)$, $\mathcal{X}_f$ and $\mathcal{X}_r$, are standard assumptions and can be found in [2], [10], [13], [14].

With the above assumptions and the formulation of the FHOCP based on approximate DTM, we can present the self-triggered strategy in the following subsection.

B. Self-Triggered MPC Strategy

In this part, we propose a self-triggered MPC strategy for the FHOCP (11). We assume that the FHOCP is solved at instant t_{k_j} and the next triggering instant is $t_{k_{j+1}}$. The optimal control and state sequence at t_{k_j} are respectively denoted as $\hat{\mathbf{u}}^*(k_j) = \{\hat{u}^*(k_j|k_j), \hat{u}^*(k_j+1|k_j), ..., \hat{u}^*(k_j+N-1|k_j)\}$ and $\hat{\mathbf{x}}^*(k_j) = \{\hat{x}^*(k_j|k_j), \hat{x}^*(k_j+1|k_j), ..., \hat{x}^*(k_j+N|k_j)\}$. The triggering instant is recursively calculated according to (5) and the value of m is determined by the condition designed as

$$m = \min\{m_1, m_2, N\} \tag{12}$$

where

$$m_1 = \sup\left\{m : \frac{e^{NLT} - e^{(N-m)LT}}{e^{LT}-1}(T\rho(T)+\eta T) \le r - \varepsilon\right\} \tag{13}$$

$$\begin{aligned} m_2 = \\ &\sup\Bigg\{\sum_{i=m}^{N-1} T\Bigg[\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})}e^{iLT}\frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\Big)^2 \\ &+ 2\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})}\Big)^2 e^{iLT} \times \\ &\qquad \frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\|\hat{x}^*(k_j+i|k_j)\|_Q\Bigg] \\ &\quad + \frac{e^{NLT}-e^{(N-m)LT}}{e^{LT}-1}(T\rho(T)+\eta T)(r+\varepsilon) \\ &\le \sigma \sum_{i=0}^{m-1} T\Big(\|\hat{x}^*(k_j+i|k_j)\|_Q^2 + \|\hat{u}^*(k_j+i|k_j)\|_P^2\Big)\Bigg\} \end{aligned} \tag{14}$$

Here, $\sigma \in (0,1)$ is defined as a performance factor. When $x(k_j) \notin \mathcal{X}_r$, the FHOCP is solved according to the proposed triggering condition. Once $x(k_j) \in \mathcal{X}_r$, a dual-mode MPC [15] is adopted. The control input which is actually applied to system (1) is

$$u(t) = \hat{u}^*(k_j+i|k_j) \tag{15}$$

over time interval $[t_{k_j+i}, t_{k_j+i+1})$, where $i = 0,1,...,k_{j+1}-k_j-1$. Algorithm 1 then summarizes the proposed approximation-based STMPC strategy.

The following subsections prove the recursive feasibility of the above strategy and the overall system stability.

C. Feasibility Analysis

The folowing lemma is used in establishing the recursive feasibility of the strategy.

***Lemma** 2 ([10]):* For $x \in \mathcal{X}_{i+m},\ y \in \mathbb{R}^n$ satisfying

$$\|x-y\|_R \le e^{iLT}\frac{e^{mLT}-1}{e^{LT}-1}(T\rho(T)+\eta T) \tag{16}$$

we have $y \in \mathcal{X}_i$.

Algorithm 1 Approximation-based STMPC strategy

Initialization: Given $j = 0$, the prediction horizon N, the sampling period T, the initial state x_0, and the related parameters $P, Q, R, \eta, \rho(T), \varepsilon, r, \sigma$.

1: If $j = 0$, go to step 3. Otherwise, go to step 2;
2: At any triggering instant $t_k = t_{k_j}$, initialize the state $\hat{x}(k_j|k_j) = x(k_j)$. If $x(k_j) \in \mathcal{X}_r$, use the local controller $\kappa(x)$. Otherwise, go to step 3;
3: Solve the FHOCP (11) to obtain a sequence of optimal control input $\hat{\mathbf{u}}^*(k_j)$;
4: Determine the value of m and the next triggering instant $t_{k_{j+1}}$ in terms of (12) and (5), respectively;
5: Apply $u(t)$ as in (15) to the actual system (1), and update the time $t_k = t_{k+1}$;
6: If $t_k = t_{k_{j+1}}$, update $j = j+1$ and go to step 2. Otherwise, go to step 5.

With Lemma 2, the feasibility analysis results is concluded in the theorem given below.

***Theorem** 1:* Consider utilizing the triggering condition (13) in system (1) with Assumption 1-3 hold, then the proposed approximation-based STMPC strategy is feasible.

Proof: It is assumed that the FHOCP (11) is solved at instant t_{k_j} and we construct a feasible control sequence $\bar{\mathbf{u}}(k_{j+1}) = \{\bar{u}(k_{j+1}|k_{j+1}), \bar{u}(k_{j+1}+1|k_{j+1}), ..., \bar{u}(k_{j+1}+N-1|k_{j+1})\}$ for the next triggering instant $t_{k_{j+1}}$ as

$$\bar{u}(k_{j+1}+i|k_{j+1}) = \begin{cases} \hat{u}^*(k_{j+1}+i|k_j) & i = 0,...,N-m-1 \\ \kappa(\bar{x}(k_{j+1}+i|k_{j+1})) & i = N-m,...,N-1 \end{cases} \tag{17}$$

and the corresponding feasible state sequence is $\bar{\mathbf{x}}(k_{j+1}) = \{\bar{x}(k_{j+1}|k_{j+1}), \bar{x}(k_{j+1}+1|k_{j+1}), ..., \bar{x}(k_{j+1}+N|k_{j+1})\}$.

The deviation between the feasible state and its predicted state satisfies

$$\begin{aligned} &\|\bar{x}(k_{j+1}+i|k_{j+1}) - \hat{x}^*(k_{j+1}+i|k_j)\|_R \\ &= \|\Gamma_T^a(\bar{x}(k_{j+1}+i-1|k_{j+1}), \hat{u}^*(k_{j+1}+i-1|k_{j+1})) \\ &\quad - \Gamma_T^a(\hat{x}^*(k_{j+1}+i-1|k_j), \hat{x}^*(k_{j+1}+i-1|k_j))\|_R \\ &\le e^{LT}\|\bar{x}(k_{j+1}+i-1|k_{j+1}) - \hat{x}^*(k_{j+1}+i-1|k_j)\|_R \\ &\le ... \\ &\le e^{iLT}\|\bar{x}(k_{j+1}|k_{j+1}) - \hat{x}^*(k_{j+1}|k_j)\|_R \\ &\le e^{iLT}\frac{e^{mLT}-1}{e^{LT}-1}(T\rho(T)+\eta T) \end{aligned} \tag{18}$$

Substituting $i = N-m$ into (18) and adopting (13) in the triggering condition make the following inequality hold

$$\begin{aligned} &\|\bar{x}(k_j+N|k_{j+1}) - \hat{x}^*(k_j+N|k_j)\|_R \\ &\qquad \le e^{(N-m)LT}\frac{e^{mLT}-1}{e^{LT}-1}(T\rho(T)+\eta T) \\ &\qquad \le r - \varepsilon \end{aligned} \tag{19}$$

We then have, by triangle inequality, $\|\bar{x}(k_j+N|k_{j+1})\|_R \le \|\hat{x}^*(k_j+N|k_j)\|_R + r - \varepsilon$. Due to $\hat{x}^*(k_j+N|k_j) \in \mathcal{X}_f$, furthermore we can obtain $\|\bar{x}(k_j+N|k_{j+1})\|_R \le r$, i.e., $\bar{x}(k_j+N|k_{j+1}) \in \mathcal{X}_r$. By applying 1) and 2) in Assumption 3, we have $\Gamma_T^a(\bar{x}(k_j+N|k_{j+1}), \kappa(\bar{x}(k_j+N|k_{j+1}))) \in \mathcal{X}_f$.

One can notice that the constructed feasible control sequence (17) will drive the states into terminal region, i.e., $\bar{x}(k_j+N+$

$1|k_{j+1}) \in \mathcal{X}_f$. Then we need to clarify that the input and state sequence corresponding to (17) satisfy the constraints in (11). Here, the expression of $\bar{\mathbf{u}}(k_{j+1})$ illustrates that the input constraints are satisfied and we shall focus on the state constraints.

It holds with Lemma 2 that

- For $i \in [0, N-m]$, due to $\hat{x}^*(k_{j+1}+i|k_j) \in \mathcal{X}_{i+m}$ and inequality (18), we can obtain $\bar{x}(k_{j+1}+i|k_{j+1}) \in \mathcal{X}_i$;
- For $i \in [N-m+1, N-1]$, due to $\bar{x}(k_j+N|k_{j+1}) \in \mathcal{X}_r$, by using the local controller $\kappa(x)$ and applying 1) and 2) in Assumption 3, we have $\bar{x}(k_{j+1}+i|k_{j+1}) \in \mathcal{X}_f \subset \mathcal{X}_i$;
- For $i = N$, due to $\bar{x}(k_{j+1}+N-1|k_{j+1}) \in \mathcal{X}_f$, the terminal constraint is obviously satisfied as $\bar{x}(k_{j+1}+N|k_{j+1}) \in \mathcal{X}_f$ by using $\kappa(x)$.

So the input and state sequence corresponding to the feasible solution (17) satisfy the constraints in (11). This completes the proof and the feasibility of the proposed strategy is established.

Moreover, the solution must exist when m reaches its minimum, i.e., $m = 1$:

$$\begin{aligned}\frac{e^{NLT} - e^{(N-1)LT}}{e^{LT}-1}&(T\rho(T)+\eta T)\\ &= e^{(N-1)LT}(T\rho(T)+\eta T) \ \leq \ r-\varepsilon\end{aligned} \tag{20}$$

In other words, the model error and the external disturbance must satisfy

$$T\rho(T)+\eta T \leq \frac{r-\varepsilon}{e^{(N-1)LT}} \tag{21}$$

D. Stability Analysis

The stability analysis results is concluded in the theorem given below.

Theorem 2: Consider utilizing the triggering condition (14) in system (1) with Assumption 1-3 hold and the initial state $x_0 \in \mathcal{X}\backslash\mathcal{X}_r$, then the system state is guaranteed to enter into $\mathcal{X}_r$ in finite time and remain in $\mathcal{X}_r$ for the future times.

Proof: Denote the optimal cost function at instant t_{k_j} as

$$J^*(k_j) = J(\hat{x}^*(k_j), \hat{\mathbf{u}}^*(k_j), N) \tag{22}$$

We apply the feasible control sequence $\bar{\mathbf{u}}(k_{j+1})$ constructed in (17). The difference of the optimal cost function between two adjacent triggering instants t_{k_j} and $t_{k_{j+1}}$ satisfies

$$\begin{aligned}&J^*(k_{j+1}) - J^*(k_j)\\ &\quad \leq J(\bar{x}(k_{j+1}), \bar{u}(k_{j+1})) - J(\hat{x}^*(k_j), \hat{u}^*(k_j))\\ &\quad = -\sum_{i=0}^{m-1} T\Big(\|\hat{x}^*(k_j+i|k_j)\|_Q^2 + \|\hat{u}^*(k_j+i|k_j)\|_P^2\Big)\\ &\quad\quad + \sum_{i=m}^{N-1} T\Big(\|\bar{x}(k_j+i|k_{j+1})\|_Q^2 - \|\hat{x}^*(k_j+i|k_j)\|_Q^2\Big)\\ &\quad\quad + \sum_{i=N}^{N+m-1} T\Big(\|\bar{x}(k_j+i|k_{j+1})\|_Q^2 + \|\bar{u}(k_j+i|k_{j+1})\|_P^2\Big)\\ &\quad\quad + \|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\hat{x}^*(k_j+N|k_j)\|_R^2\end{aligned} \tag{23}$$

Substituting (18) into (23) and applying square difference formula yield

$$\begin{aligned}&\sum_{i=m}^{N-1} T(\|\bar{x}(k_j+i|k_{j+1})\|_Q^2 - \|\hat{x}^*(k_j+i|k_j)\|_Q^2)\\ &\quad = \sum_{i=0}^{N-m-1} T(\|\bar{x}(k_{j+1}+i|k_{j+1})\|_Q^2 - \|\hat{x}^*(k_{j+1}+i|k_j)\|_Q^2)\\ &\quad \leq \sum_{i=0}^{N-m-1} T\Bigg[\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})} e^{iLT}\frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\Big)^2\\ &\quad\quad + 2\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})}\Big)^2 e^{iLT}\times\\ &\quad\quad\quad \frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\|\hat{x}^*(k_{j+1}+i|k_j)\|_Q\Bigg]\end{aligned} \tag{24}$$

The last two terms of inequality (23) can be rewritten as

$$\begin{aligned}&\|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\hat{x}^*(k_j+N|k_j)\|_R^2\\ &\quad = \|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\bar{x}(k_j+N|k_{j+1})\|_R^2\\ &\quad\quad + \|\bar{x}(k_j+N|k_{j+1})\|_R^2 - \|\hat{x}^*(k_j+N|k_j)\|_R^2\end{aligned} \tag{25}$$

With 3) in Assumption 3, we have

$$\begin{aligned}&\|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\bar{x}(k_{j+1}+N-1|k_{j+1})\|_R^2\\ &\quad \leq -T\big(\|\bar{x}(k_{j+1}+N-1|k_{j+1})\|_Q^2\\ &\quad\quad + \|\bar{u}(k_{j+1}+N-1|k_{j+1})\|_P^2\big)\\ &\|\bar{x}(k_{j+1}+N-1|k_{j+1})\|_R^2 - \|\bar{x}(k_{j+1}+N-2|k_{j+1})\|_R^2\\ &\quad \leq -T\big(\|\bar{x}(k_{j+1}+N-2|k_{j+1})\|_Q^2\\ &\quad\quad + \|\bar{u}(k_{j+1}+N-2|k_{j+1})\|_P^2\big)\\ &\vdots\\ &\|\bar{x}(k_j+N+1|k_{j+1})\|_R^2 - \|\bar{x}(k_j+N|k_{j+1})\|_R^2\\ &\quad \leq -T\big(\|\bar{x}(k_j+N|k_{j+1})\|_Q^2\\ &\quad\quad + \|\bar{u}(k_j+N|k_{j+1})\|_P^2\big)\end{aligned} \tag{26}$$

By summing up the inequalities given above, one can obtain

$$\begin{aligned}&\sum_{i=N}^{N+m-1} T\Big(\|\bar{x}(k_j+i|k_{j+1})\|_Q^2 + \|\bar{u}(k_j+i|k_{j+1})\|_P^2\Big)\\ &+\|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\bar{x}(k_j+N|k_{j+1})\|_R^2 \leq 0\end{aligned} \tag{27}$$

So the last three terms of inequality (23) satisfy

$$\begin{aligned}&\sum_{i=N}^{N+m-1} T\Big(\|\bar{x}(k_j+i|k_{j+1})\|_Q^2 + \|\bar{u}(k_j+i|k_{j+1})\|_P^2\Big)\\ &+ \|\bar{x}(k_{j+1}+N|k_{j+1})\|_R^2 - \|\hat{x}^*(k_j+N|k_j)\|_R^2\\ &\quad \leq \|\bar{x}(k_j+N|k_{j+1})\|_R^2 - \|\hat{x}^*(k_j+N|k_j)\|_R^2\\ &\quad \leq (\|\bar{x}(k_j+N|k_{j+1})\|_R - \|\hat{x}^*(k_j+N|k_j)\|_R)\times\\ &\quad\quad\quad (\|\bar{x}(k_j+N|k_{j+1})\|_R + \|\hat{x}^*(k_j+N|k_j)\|_R)\\ &\quad \leq \frac{e^{NLT}-e^{(N-m)LT}}{e^{LT}-1}(T\rho(T)+\eta T)(r+\varepsilon)\end{aligned} \tag{28}$$

Substituting inequalities (24) and (28) into (23) obtains

$$
\begin{aligned}
&J^*(k_{j+1}) - J^*(k_j) \\
&\le -\sum_{i=0}^{m-1} T\Big(\|\hat{x}^*(k_j+i|k_j)\|_Q^2 + \|\hat{u}^*(k_j+i|k_j)\|_P^2\Big) \\
&\quad + \sum_{i=m}^{N-1} T\Bigg[\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})} e^{iLT} \frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\Big)^2 \\
&\quad + 2\Big(\frac{\overline{\lambda}(\sqrt{Q})}{\underline{\lambda}(\sqrt{R})}\Big)^2 e^{iLT} \times \\
&\qquad \frac{e^{mLT}-1}{e^{LT}-1}\big(T\rho(T)+\eta T\big)\|\hat{x}^*(k_j+i|k_j)\|_Q\Bigg] \\
&\quad + \frac{e^{NLT}-e^{(N-m)LT}}{e^{LT}-1}\big(T\rho(T)+\eta T\big)(r+\varepsilon)
\end{aligned} \tag{29}
$$

Adopting (14) in the triggering condition makes the following inequality hold:

$$
\begin{aligned}
&J^*(k_{j+1}) - J^*(k_j) \le -(1-\sigma)\times \\
&\quad \sum_{i=0}^{m-1} T\Big(\|\hat{x}^*(k_j+i|k_j)\|_Q^2 + \|\hat{u}^*(k_j+i|k_j)\|_P^2\Big) < 0
\end{aligned} \tag{30}
$$

If there exists a system state which is out of $\mathcal{X}_r$ all the time, then $J^*(k_j) \to -\infty$ as $k_j \to \infty$, contradicting with $J^*(k_j) \ge 0$. So the inequality (30) ensures that the system state will enter into $\mathcal{X}_r$ in finite time. The property 3) in Assumption 3 and the condition in (21) guarantee that the system state will remain in $\mathcal{X}_r$. The proof is then completed.

IV. Simulation Results

In this part, we consider a cart-damper-spring system discussed in [16] as follows:

$$
\begin{cases}
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = -\frac{k}{M_c} e^{-x_1(t)} x_1(t) - \frac{h_d}{M_c} x_2(t) + \frac{u(t)}{M_c} + w(t)
\end{cases} \tag{31}
$$

where $x_1(t)$ and $x_2(t)$ represent the cart's displacement and velocity respectively, $u(t)$ is the control input, $w(t)$ is the external disturbance, M_c, k and h_d denote the mass of the cart, the factor of the spring and the damping factor respectively. The plant states and the control input are constrained by $|x_1(t)| \le 1$m, $|x_2(t)| \le 1$m/s and $|u(t)| \le 0.5$N. The external disturbance is bounded by $\eta = 0.005$. The other related parameters are given as follows: $M_c = 0.75$kg, $k = 0.12$N/m, $h_d = 0.6$N·s/m. For simplicity, using the disturbance-free Euler approximation of the system (31) as

$$
\begin{cases}
x_1(k+1) = x_1(k) + T x_2(k) \\
x_2(k+1) = -\frac{kT}{M_c} e^{-x_1(k)} x_1(k) + (1 - \frac{Th_d}{M_c}) x_2(k) + \frac{Tu(k)}{M_c}
\end{cases} \tag{32}
$$

where the sampling period is determined as $T = 0.1$s. Other forms of approximation: modified Euler approximation and Runge-Kutta approximation, to name a few, have been established in [17].

Consider the implementation of the STMPC strategy into the approximate DTM (32), the prediction horizon is set as $N = 5$. The Lipschitz constant in (3) is calculated as $L = 1.5297$. We select the form of the model error in (8) as $T\rho(T) = LM_cT^2/2$. The limits in $\mathcal{X}_f$ and $\mathcal{X}_r$ are calculated as $\varepsilon = 0.5814$ and $r = 0.6058$. The two weighting matrices P and Q are set as $P = [1]$ and $Q = [1\ 0; 0\ 1]$. Then the weighting matrix R can be determined as $R = [3.2602\ 1.1599; 1.1599\ 1.4811]$. The initial state is $x_0 = [-0.7\ 0.8]^T$. With MATLAB function `fmincon`, the FHOCP in (11) can be solved then.

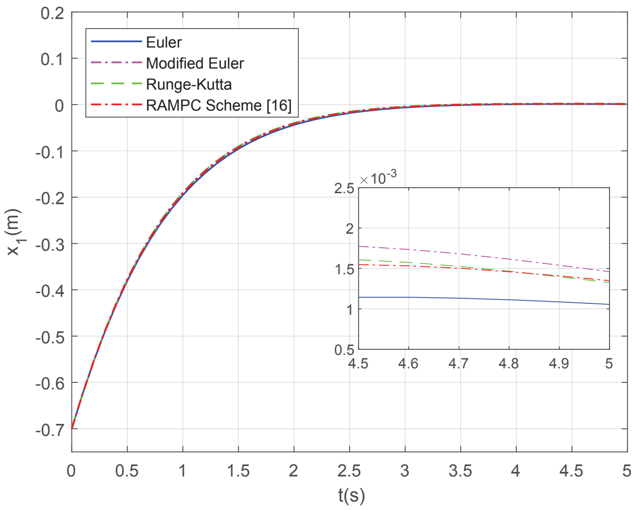

Fig. 1. Trajectories of system state x_1 under the proposed approximation-based STMPC strategy and the RAMPC strategy in [16].

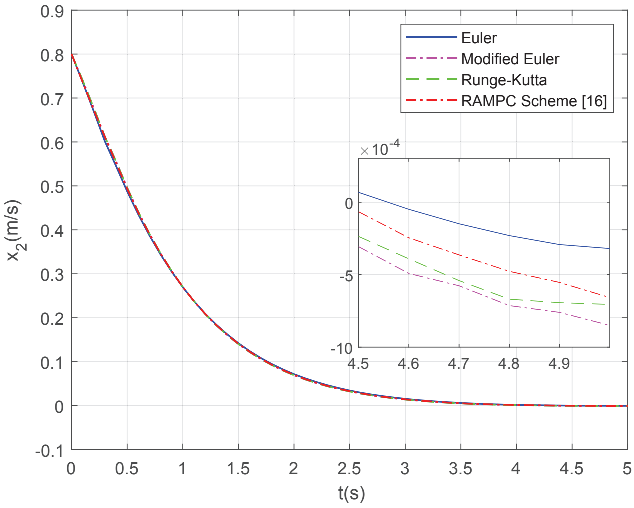

Fig. 2. Trajectories of system state x_2.

The simulation results are given in Fig. 1-4. by adopting three general approximations: Euler approximation, modified Euler approximation and Runge-Kutta approximation. To show the effectiveness of the approximation-based STMPC strategy, we compare our results with the robust approximation-based MPC (RAMPC) strategy in [16], i.e., using periodic sampling. It can be seen from Fig. 1-3. that the continuous-time state and input constraints are satisfied. Moreover, the state trajectories with bounded disturbance under the proposed strategy converge to a neighbourhood of the origin and are similar to the RAMPC strategy, showing that our strategy has comparable performance. One can notice from Fig. 4. that the proposed approximation-based STMPC strategy reduces the computing frequency significantly compared with the RAMPC strategy.

V. Conclusion

In this paper, we investigate a self-triggered MPC problem for perturbed constrained nonlinear sampled-data systems. An approximation-based STMPC strategy which ensures the feasibility of the control problem and the stability of the overall system has been put forward. The proposed strategy adopts approximate DTM of the continuous-time system, making the computation of the FHOCP tractable. By designing the triggering condition, the

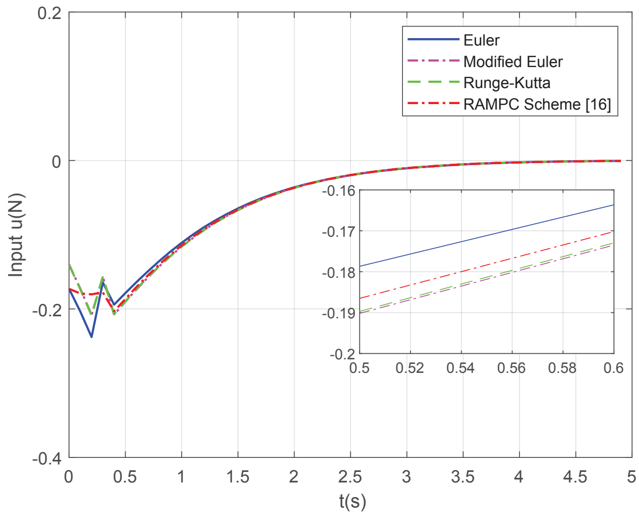

Fig. 3. Trajectories of control input u.

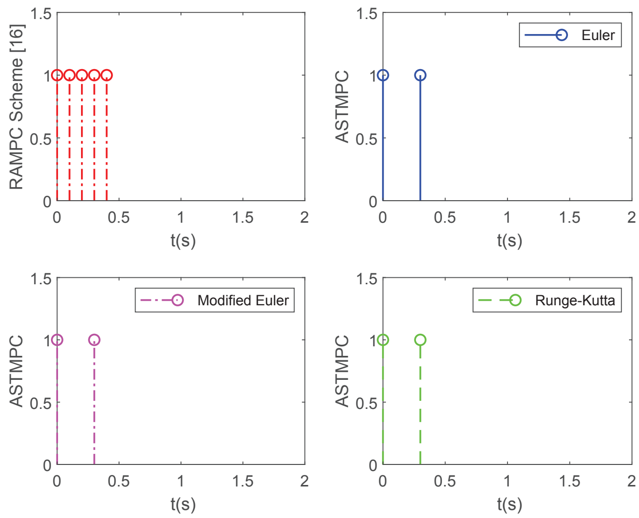

Fig. 4. Triggering instants under two strategies.

triggering instants to solve the control problem are pre-determined and the computation load is alleviated. Eventually, the effectiveness of our theoretical strategy is verified by a numerical simulation.

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